

TOWARDS ON-LINE OHBA'S CONJECTURE

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ABSTRACT. Ohba conjectured that every graph G with $|V(G)| \leq 2\chi(G)+1$ has its choice number equal its chromatic number. The on-line choice number of a graph is a variation of the choice number defined through a two person game, and is always at least as large as its choice number. Based on the result that for $k \geq 3$, the complete multipartite graph $K_{2*(k-1),3}$ is not on-line k -choosable, the on-line version of Ohba's conjecture is modified in [P. Huang, T. Wong and X. Zhu, Application of polynomial method to on-line colouring of graphs, European J. Combin., 2011] as follows: Every graph G with $|V(G)| \leq 2\chi(G)$ has its on-line choice number equal its chromatic number. In this paper, we prove that for any graph G , there is an integer n such that the join $G + K_n$ of G and K_n has its on-line choice number equal chromatic number. Then we show that the on-line version of Ohba conjecture is true if G has independence number at most 3. We also present an alternative proof of the result that Ohba's conjecture is true for graphs of independence number at most 3 and an alternative proof of the following result of Kierstead: For any positive integer k , the complete multipartite graph K_{3*k} has choice number $\lceil (4k-1)/3 \rceil$. Finally, we prove that the on-line choice number of K_{3*k} is at most $\frac{3}{2}k$. The exact value of the on-line choice number of K_{3*k} remains unknown.

1. INTRODUCTION

A *list assignment* of a graph G is a mapping L which assigns to each vertex v a set $L(v)$ of permissible colours. An L -*colouring* of G is a proper vertex colouring of G which colours each vertex with one of its permissible colours. We say that G is L -*colourable* if there exists an L -colouring of G . A graph G is called k -*choosable* if for any list assignment L with $|L(v)| = k$, for all $v \in V(G)$, G is L -colourable. More generally, for a function $f : V(G) \rightarrow \mathbb{N}$, we say G is f -*choosable* if for every list assignment L with $|L(v)| = f(v)$, G is L -colourable. The *choice number* $\text{ch}(G)$ of G is the minimum k for which G is k -choosable. List colouring of graphs has been studied extensively in the literature [21, 3, 20].

A list assignment of a graph G can be given alternatively as follows: Without loss of generality, we may assume that $\cup_{v \in V(G)} L(v) = \{1, 2, \dots, q\}$ for some integer q . For $i = 1, 2, \dots, q$, let $V_i = \{v : i \in L(v)\}$. The sequence (V_1, V_2, \dots, V_q) is another way of specifying the list assignment. An L -colouring of G is equivalent to a sequence (X_1, X_2, \dots, X_q) of independent sets that form a partition of $V(G)$ and such that $X_i \subseteq V_i$ for $i = 1, 2, \dots, q$. This point of view of list colouring motivates the definition of the following list colouring game on a graph G , which was introduced in [18, 17].

Definition. Given a finite graph G and a mapping $f : V(G) \rightarrow \mathbb{N}$, two players play the following game. In the i -th step, Player A chooses a non-empty subset V_i of $V(G)$, and Player B chooses an independent set X_i contained in V_i . A vertex v is

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coloured before the i th step if $v \in X_j$ for some $j < i$, and is finished before the i th step if v is contained in $f(v)$ of the V_j 's with $j < i$. When Player A chooses the set V_i , it is required V_i contains only uncoloured non-finished vertices. If for some integer m , before the m -th step, there is a finished vertex v that is uncoloured, then Player A wins the game. Otherwise, at some step, all vertices are coloured. In this case, Player B wins the game.

We call such a game the *on-line (G, f) -list colouring game*. We say G is *on-line f -choosable* if Player B has a winning strategy in the on-line (G, f) -list colouring game, and we say G is *on-line k -choosable* if G is on-line f -choosable for the constant function $f \equiv k$. The *on-line choice number* of G , denoted by $\text{ch}^{\text{OL}}(G)$, is the minimum k for which G is on-line k -choosable.

It follows from the definition that for any graph G , $\text{ch}^{\text{OL}}(G) \geq \text{ch}(G)$. There are graphs G with $\text{ch}^{\text{OL}}(G) > \text{ch}(G)$ (see [22]). It remains a challenging open problem whether the difference $\text{ch}^{\text{OL}}(G) - \text{ch}(G)$ can be arbitrarily large. Alon [1] proved that if $\text{ch}(G) \leq k$ then its colouring number $\text{col}(G)$ is at most $f(k) = 4 \binom{k^4}{s} \log_2(2 \binom{k^4}{k})$. This gives us an exponential bound for the on-line choice number of G in terms of its choice number

$$f(\text{ch}(G)) \geq \text{col}(G) \geq \text{ch}^{\text{OL}}(G).$$

Many currently known upper bounds for the choice number of a graph remain upper bounds for its on-line choice number. For example, the on-line choice number of planar graphs is at most 5 [17], the on-line choice number of planar graphs of girth at least 5 is at most 3 [17, 2], the on-line choice number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$ [17], and if G has an orientation in which the number of even eulerian subgraphs differs from the number of odd eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is on-line f -choosable [18].

A graph G is called *chromatic-choosable* (respectively, *on-line chromatic-choosable*) if $\chi(G) = \text{ch}(G)$ (respectively, $\chi(G) = \text{ch}^{\text{OL}}(G)$). The problem which graphs are chromatic-choosable has been extensively studied. A few well-known classes of graphs are conjectured to be chromatic-choosable. These include line graphs (conjectured independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris, see [6] and [9]), claw-free graphs [5], and square of graphs [13], etc. It is proved by Galvin [4] that the line graph of a bipartite graph is always chromatic-choosable. As observed by Schauz [17], the same proof works for on-line list colouring as well. So the line graph of a bipartite graph is on-line chromatic-choosable. In this paper, we are interested in Ohba's conjecture [14], which also concerns chromatic-choosable graphs.

Conjecture 1 (Ohba 2002). *If $|V(G)| \leq 2\chi(G) + 1$, then $\chi(G) = \text{ch}(G)$.*

Some special cases of Ohba's conjecture are already verified. Reed and Sudakov [16, 15] proved that it holds for all graphs G with $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{3}$ and soon afterwards they gave an asymptotic-type result that for any $\varepsilon > 0$ there is an integer n_0 such that all graphs with $n_0 \leq |V(G)| \leq (2 - \varepsilon)\chi(G)$ are chromatic-choosable. Recently, Kostochka et al. (see [12]) proved that Conjecture 1 holds for all graphs with independence number at most 5 which improves the results of [7, 19].

Note that it suffices to consider the conjecture only for complete multipartite graphs. Suppose $k = k_1 + k_2 + \dots + k_s$, and n_1, n_2, \dots, n_s are positive integers. We denote by $K_{n_1 \star k_1, n_2 \star k_2, \dots, n_s \star k_s}$ the complete k -partite graph in which k_i parts are of cardinality n_i for $i = 1, 2, \dots, s$. If $k_i = 1$, then $n_i \star 1$ in the subscript will be shortened as n_i (for example $K_{3, 2 \star 3} = K_{3 \star 1, 2 \star 3}$).

It is proved in [11] that for $k \geq 2$, the graph $K_{3, 2 \star k}$ is not on-line $(k+1)$ -choosable. However, experiments and preliminary results show that a slightly modified version

of Ohba's conjecture might be true in the on-line case. The following conjecture is proposed in [8].

Conjecture 2. *If $|V(G)| \leq 2\chi(G)$, then $\chi(G) = \text{ch}^{\text{OL}}(G)$.*

The on-line version of Ohba's conjecture seems to be more difficult to handle. Some of the key technique used in the study of Ohba's conjecture do not apply to the on-line version. For example, it is easy to prove that $K_{2\star k}$ is k -choosable. However, all the previously known proofs of this result use Hall Theorem, and this cannot be directly applied to the on-line version. In [8], the method of Combinatorial Nullstellensatz is used to prove that $K_{2\star k}$ is k -choosable. By a result of Schauz mentioned above, this implies that $K_{2\star k}$ is on-line k -choosable. Recently, a simple strategy was given in [11] for Player B to win this on-line (G, f) -colouring game.

By using Combinatorial Nullstellensatz, $K_{\ell+1, 1\star(\ell-1), 2\star(k-\ell)}$, $K_{s, t, 1\star(k-2)}$ (where $(s-1)(t-1) \leq 2k-3$), $K_{3\star 2, 1\star 2, 2\star(k-4)}$ and $K_{4, 3, 1\star 3, 2\star(k-5)}$ are shown in [8] to be on-line k -choosable. Still, we know much less about Conjecture 2 than about Conjecture 1.

The main focus of this paper is the on-line version of Ohba's conjecture. We prove that for any graph G , by adding enough universal vertices, the resulting graph is on-line chromatic-choosable. I.e., for a sufficiently large integer n , the join $G + K_n$ of G and K_n is on-line chromatic-choosable. In fact the argument gives that $\chi(G) = \text{ch}^{\text{OL}}(G)$ for all graphs G with $|V(G)| \leq \chi(G) + \sqrt{\chi(G)}$. Then we prove that Conjecture 2 is true for graphs with independence number at most 3, and also give an alternate proof of the result that Conjecture 1 is true for graphs with independence number at most 3.

We finish with the discussion on the choice number and on-line choice number of $K_{3\star k}$. These graphs are natural candidates to prove a hypothetic separation (by more than a constant) of choice number and on-line choice number. With an ingenious argument, Kierstead proved in [10] that $\text{ch}(K_{3\star k}) \leq \lceil (4k-1)/3 \rceil$. This result matches the lower bound given by Erdős, Rubin and Taylor [3]. We prove that $\text{ch}^{\text{OL}}(K_{3\star k}) \leq \frac{3}{2}k$, and present an alternative proof of Kierstead's result.

2. THE JOIN OF G AND K_n

We are going to prove here that for any graph G , by adding enough universal vertices, one can construct a graph that is on-line chromatic-choosable. For two graphs G and G' , the *join* of G and G' , denoted by $G + G'$ is the graph obtained from the disjoint union of G and G' by adding all the possible edges between $V(G)$ and $V(G')$.

Theorem 3. *For every graph G there exists a positive integer n such that $\chi(G + K_n) = \text{ch}^{\text{OL}}(G + K_n)$.*

Proof. Without loss of generality, we may assume that G is a complete $\chi(G)$ -partite graph. Let us start with an easy observation (see [17]): Assume H is a graph and $f : V(H) \rightarrow \mathbb{N}$ is a function. If $f(v) \geq d_H(v) + 1$ for a vertex $v \in V(H)$, then

H is on-line f -choosable if and only if $H - v$ is on-line f -choosable.

For a given graph G , we put $H_0 = G + K_n$ with $n = |V(G)|^2$ and $f(v) = \chi(H_0) = \chi(G) + n$ for all $v \in V(H_0)$. Let $V_1, \dots, V_{\chi(H_0)}$ be a partition of $V(H_0)$ into independent sets. We are going to present a winning strategy for Player B in the on-line (H_0, f) -list colouring game.

We denote by H_i a subgraph of all uncoloured vertices of H_0 after i steps. Before playing the $(i+1)$ -th step, we delete from H_i , one by one, all the vertices v with $f(v) \geq d_{H_i}(v) + 1$ (by using the observation above). The resulting graph is still

denoted by H_i . Now, by a *part* of H_i we mean a non-empty set of the form $V_j \cap H_i$ for $1 \leq j \leq \chi(H_0)$. Assume at the $(i+1)$ th step, Player A chooses a subset U_i . Player B finds an independent set I contained in U_i according to the following algorithm.

Algorithm 1: Strategy for Player B in the $(i+1)$ -th step

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1  if there is a part  $V$  of  $H_i$  with  $|V| \geq 2$  and  $V \subseteq U_i$  then
2    pick  $I = V$ 
3  else if there is a part  $V$  of  $H_i$  with  $|V| = 1$  and  $V \subseteq U_i$  then
4    pick  $I = V$ 
5  else
6    pick  $I$  to be any maximal independent set in  $U_i$ 

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For $v \in V(G \cap H_i)$, let $f_i(v)$ be the number of remaining colours for v just before the $(i+1)$ th step, and define the *deficit* of v as $d_{H_i}(v) + 1 - f_i(v)$, which is the number of additional colours needed so that v can be removed from the graph (by the observation we started with). Since vertices v with $f_i(v) \geq d_{H_i}(v) + 1$ are removed, we know that the deficit of each vertex v is positive. The deficit of a part V of H_i is the sum of deficits of its vertices

$$\sum_{v \in V} (d_{H_i}(v) + 1 - f_i(v)).$$

We will show that after every step of the game the deficit of each part of size at least 2 decreases. Let V be a part of H_i and $|V| \geq 2$.

If line 2 is executed, then either part V is picked and it disappears in H_{i+1} , or $d_{H_{i+1}}(v) \leq d_{H_i}(v) - 2$ and $f_{i+1}(v) \geq f_i(v) - 1$ for all $v \in V$. Hence the deficit of each vertex of V decreases.

If line 4 is executed, then $d_{H_{i+1}}(v) = d_{H_i}(v) - 1$, $f_{i+1}(v) \geq f_i(v) - 1$ for all $v \in V$, and there exists $v \in V$ with $f_{i+1}(v) = f_i(v)$ as V is not contained in U_{i+1} . So the total deficit of V decreases.

Assume line 6 is executed. If $I = V \cap U_{i+1}$, then $d_{H_{i+1}}(v) = d_{H_i}(v)$, $f_{i+1}(v) = f_i(v)$ for all $v \in V - U_{i+1}$ so the sum decreases as the deficit of erased vertices is positive. Otherwise, $d_{H_{i+1}}(v) \leq d_{H_i}(v) - 1$ and $f_{i+1}(v) \geq f_i(v) - 1$ for all $v \in V$ and there exists $v \in V$ with $f_{i+1}(v) = f_i(v)$. So the deficit of V decreases.

As each $v \in V(H_0)$ has deficit bounded by $|V(G)|$, each part has initially deficit bounded by $n = |V(G)|^2$. Since after each step the deficit of each part of size at least 2 decreases and vertices with non-positive deficit are deleted, after n rounds the remaining graph, namely H_n , forms a clique.

The vertices in H_n may come from G or K_n and there are at most $\chi(G)$ vertices coming from G , at most one for each part of G . If $U_i \cap K_n \neq \emptyset$ then the number of parts in H_{i+1} decreases by 1 comparing to the number of parts in H_i (as line 2 or 4 is executed). Therefore

$$f_n(v) \geq \text{the number of parts in } H_n = d_{H_n}(v) + 1 \quad \text{for all } v \in H_n \cap K_n$$

For vertices $v \in H_n \cap G$, as each step decreases the number of permissible colours by at most 1, we have $f_n(v) \geq f_0(v) - n = \chi(G)$. By applying the observation repeatedly, these inequalities certify that all vertices of H_q are removed and H_q is empty, which finishes the proof. \square

The argument presented gives also an Ohba-like statement with much more restricted constraint on the size and the chromatic number of a graph.

Corollary 4. *If $|V(G)| \leq \chi(G) + \sqrt{\chi(G)}$, then $\chi(G) = \text{ch}^{\text{OL}}(G)$.*

3. A LEMMA

In the remainder of this paper, we consider complete multipartite graphs of independence number at most 3, i.e., graphs of the form $K_{3 \star k_3, 2 \star k_2, 1 \star k_1}$ for some integers $k_1, k_2, k_3 \geq 0$. Lemma 5 below specifies a sufficient condition for such a graph G to be on-line f -choosable. In further sections we are going to derive from this a few quite independent results.

For a subset U of $V(G)$, let $\delta_U : V(G) \rightarrow \{0, 1\}$ be the characteristic function of U , i.e., $\delta_U(x) = 1$ if $x \in U$ and $\delta_U(x) = 0$ otherwise. The following observation follows directly from the definition of the on-line (G, f) -colouring game (see [17]).

Observation. *If G is an edgeless graph and $f(v) \geq 1$ for all $v \in V(G)$, then G is on-line f -choosable. If G has at least one edge, then G is on-line f -choosable if and only if for every $U \subseteq V(G)$, there is an independent set I of G such that $I \subseteq U$ and $G - I$ is on-line $(f - \delta_U)$ -choosable.*

Lemma 5. *Let G be a complete multipartite graph G with each part of size at most 3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$ be a partition of the set of parts of G into classes such that \mathcal{A} contains only parts of size 1, \mathcal{B} contains only parts of size 2, \mathcal{C} contains only parts of size 3 and \mathcal{S} contains parts of size 1 or 2. Let k_1, k_2, k_3, s denote the cardinalities of classes $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$, respectively. Suppose that classes \mathcal{A} and \mathcal{S} are ordered i.e. $\mathcal{A} = (A_1, \dots, A_{k_1})$ and $\mathcal{S} = (S_1, \dots, S_s)$. For $1 \leq i \leq s$, let $v_S(i) = \sum_{1 \leq j < i} |S_j| + 1$. Assume $f : V(G) \rightarrow \mathbb{N}$ is a function for which the following conditions hold*

$$f(v) \geq k_3 + k_2 + i, \quad \text{for all } 1 \leq i \leq k_1 \text{ and } v \in A_i \quad (1)$$

$$f(v) \geq 2k_3 + k_2 + k_1 + v_S(i), \quad \text{for all } 1 \leq i \leq s \text{ and } v \in S_i \quad (1')$$

$$f(v) \geq k_3 + k_2, \quad \text{for all } v \in B \in \mathcal{B} \quad (2.1)$$

$$\sum_{v \in B} f(v) \geq |V(G)|, \quad \text{for all } B \in \mathcal{B} \quad (2.2)$$

$$f(v) \geq k_3 + k_2, \quad \text{for all } v \in C \in \mathcal{C} \quad (3.1)$$

$$f(u) + f(v) \geq |V(G)| - 1, \quad \text{for all } u, v \in C \in \mathcal{C}, u \neq v \quad (3.2)$$

$$\sum_{v \in C} f(v) \geq |V(G)| - 1 + k_3 + k_2 + k_1, \quad \text{for all } C \in \mathcal{C}. \quad (3.3)$$

Then G is on-line f -choosable.

Proof. The proof goes by induction on $|V(G)|$. If G is edgeless, i.e., $k_1 + k_2 + k_3 + s = 1$, then G is on-line f -choosable as $f(v) \geq 1$ for all $v \in V(G)$. Assume now that G has at least two parts and that the statement is verified for all smaller graphs.

Given $U \subseteq V(G)$, we shall find an independent set I of G such that $I \subseteq U$ and $G - I$ is on-line $(f - \delta_U)$ -choosable. Let $G' = G - I$ and $f' = f - \delta_U$. Note that $f'(v) \geq f(v) - 1$ for all $v \in V(G)$. Clearly, G' is also a complete multipartite graph with each part of size at most 3. We are going to show that G' with f' , an appropriate partition $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{S}'$ and orderings of \mathcal{A}' and \mathcal{S}' fulfill the conditions of Lemma 5. Hence, by induction hypothesis G' is on-line f' -choosable.

The strategy of choosing an independent set I is given by the case distinction. Note that we consider the setting of Case i only when the conditions for all $i - 1$ previous cases do not hold. When we verify the inequalities from the statement of Lemma 5 for G' and f' we usually compare the total decrease/increase of left and right hand sides with the analogous inequalities that hold for G and f . The notation for the parts of G' and its sizes is analogous as for G , e.g. A'_i, S'_i, k'_1, s' and so on. Partition $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{S}'$ and orders on the classes \mathcal{A}' and \mathcal{S}' are usually inherited. In the case distinction below we comment the partitions only if the order or partition changes in the considered step.

Case 1. $C \subseteq U$ for some $C \in \mathcal{C}$.

Put $I = C$. Then $k'_3 = k_3 - 1$ and all other parameters remain the same. Note that $|V(G')| = |V(G)| - 3$. Now, it is immediate that G' with inherited partition and f' satisfy the conditions of Lemma 5.

Case 2. $B \subseteq U$ for some $B \in \mathcal{B}$.

Put $I = B$. Then $k'_2 = k_2 - 1$ and all other parameters remain the same. Note that $|V(G')| = |V(G)| - 2$. Again, it is immediate that G' with inherited partition and f' satisfy the conditions of Lemma 5. Note that because Case 1 does not apply, for inequality (3.3), the left-hand side decreases by at most 2.

In all remaining cases, as conditions for cases 1 and 2 do not hold, we have

- (i) U covers at most one vertex in each $B \in \mathcal{B}$ (we are not in Case 2). This implies that inequalities (2.2) for any G' will trivially hold provided $|V(G')| \leq |V(G)| - 1$.
- (ii) U covers at most two vertices in each $C \in \mathcal{C}$ (we are not in Case 1).

Case 3. There is $C \in \mathcal{C}$ with $U \cap C = \{u, v\}$ and (3.1) is saturated for v or (3.2) is saturated for u and v .

Let $C = \{u, v, w\}$. Put $I = \{u, v\}$. Then $k'_3 = k_3 - 1$, $k'_1 = k_1 + 1$ and all other parameters remain unchanged. Indeed, we colour two vertices of C and the remaining vertex forms $A'_{k'_1} = \{w\}$, a new part of size 1, which is appended to the ordering of \mathcal{A}' . Note that $|V(G')| = |V(G)| - 2$.

Now, we need to check that all the inequalities of Lemma 5 hold for G' and f' . Inequality (1) holds for A'_i with $1 \leq i < k'_1$ as the right hand side decreases by 1 and the left hand side decreases at most by 1. Inequality (1) holds for $A'_{k'_1} = \{w\}$ either because (3.2) is saturated for u, v in G and hence

$$f'(w) = f(w) \geq |V(G)| - 1 + k_3 + k_2 + k_1 - (|V(G)| - 1) = k'_3 + k'_2 + k'_1,$$

or because (3.1) is saturated for v in G and hence

$$f'(w) = f(w) \geq |V(G)| - 1 - k_3 - k_2 \geq 2k_3 + k_2 + k_1 - 1 = 2(k'_3 + 1) + k'_2 + (k'_1 - 1) - 1.$$

The inequality (3.3) for $C \in \mathcal{C}$ holds as the right hand side decreased by 2 and the left hand side decreased by at most 2 (see (ii)). The other inequalities hold trivially.

Note that in all remaining cases

- (i) For each $C \in \mathcal{C}$ either $|U \cap C| \leq 1$, or $|U \cap C| = 2$ and (3.2) is not saturated for $U \cap C$ in G (we are not in Case 3). This implies that inequalities (3.2) will hold for any G' provided $|V(G')| \leq |V(G)| - 1$.

Case 4. There is $B \in \mathcal{B}$ with $U \cap B = \{v\}$ and (2.1) is saturated for v .

Let $B = \{u, v\}$. Put $I = \{v\}$. Then $k'_2 = k_2 - 1$ and $s' = s + 1$ and all other parameters remain unchanged. The part $\{u\}$ form a new part of size 1 and is appended at the end of the order to the class \mathcal{S} as $S'_{s'}$. Note that $|V(G')| = |V(G)| - 1$.

We are going to check the inequalities for G' and f' . Inequalities (1) for A'_j with $1 \leq j \leq k'_1$ and (1') for S'_j with $1 \leq j \leq s' - 1$ hold as the right hand side decreases by 1 while the left hand side decreases at most by 1. Inequality (1') for $S'_{s'} = \{u\}$ holds by (2.2) for u, v in G and the saturation of (2.1) for v in G

$$f'(u) = f(u) \geq |V(G)| - k_3 - k_2 = 2k'_3 + (k'_2 + 1) + k_1 + (v_{S'}(s') - 1).$$

The inequalities (2.1), (3.1) and (3.3) for G' with f' hold trivially.

Note that in all remaining cases

- (i) For all $v \in U \cap \bigcup_{B \in \mathcal{B}} B$ the inequality (2.1) is not saturated for v in G . This means that (2.1) will hold in any G' .

Case 5. There is $C \in \mathcal{C}$ with $U \cap C = \{v\}$ and (3.1) is saturated for v .

Let $C = \{u, v, w\}$ and put $I = \{v\}$. The remaining part $\{u, w\}$ is appended at the end of the sequence \mathcal{S} . Note that $|V(G')| = |V(G)| - 1$.

The inequalities (1) for A'_j with $1 \leq j \leq k'_1$ and (1') for S'_j with $1 \leq j \leq s-1$ hold as the right hand side decreases by 1 while the left hand side decreases at most by 1. Inequalities (1') for the vertices of the new part $S'_{s'} = \{u, w\}$ hold because (3.1) is saturated for v in G and hence for $x \in \{u, w\}$,

$$f'(x) = f(x) \geq (|V(G)| - 1) - k_3 - k_2 = 2(k'_3 + 1) + k'_2 + k'_1 + (v_{S'}(s') - 1) - 1,$$

The inequalities (2.1), (3.1) for G' are trivial. The inequalities (3.3) for G' hold as the right hand side decreases by 2 and the left hand side at most by 2 (see (ii)).

Note that in all remaining cases

- (i) For all $v \in U \cap \bigcup_{C \in \mathcal{C}} C$ the inequality (3.1) is not saturated for v in G . This means that (3.1) will hold in any G' .

Case 6. There is $1 \leq i \leq k_1$ with $A_i \subseteq U$.

Let i be the least index with $A_i = \{v\} \subseteq U$. Put $I = \{v\}$. Then $k'_1 = k_1 - 1$ and all other parameters remain unchanged. We also renumber the parts of size 1, namely $A'_j = A_{j+1}$ for $i \leq j \leq k'_1$. Note that $|V(G')| = |V(G)| - 1$.

The inequality (1) for A'_j with $1 \leq j < i$ holds as both sides are the same in G' as in G . The inequality (1) for $A'_j = A_{j+1} = \{u\}$ with $i \leq j \leq k'_1$ holds as

$$f'(u) \geq f(u) - 1 \geq k_3 + k_2 + (j + 1) - 1.$$

The inequalities (3.3) hold in G' as the right hand side decreased by 2 and the left hand side decreased by at most 2 (see (ii)).

Case 7. There is $C \in \mathcal{C}$ with $U \cap C = \{u, v\}$.

Let $C = \{u, v, w\}$. Put $I = \{u, v\}$. Then $k'_3 = k_3 - 1$ and $k'_1 = k_1 + 1$ and all other parameters remain unchanged. There is one new part of size 1, namely $A'_1 = \{w\}$, and all the others are renumbered $A'_j = A_{j-1}$ for $2 \leq j \leq k'_1$. Note that $|V(G')| = |V(G)| - 1$.

The inequality (1) for $A'_1 = \{w\}$ holds by (3.1) for w in G

$$f'(w) = f(w) \geq k_3 + k_2 = (k'_3 + 1) + k'_2.$$

The inequality (1) for $A'_j = A_{j-1} = \{x\}$ with $2 \leq j \leq k'_1$ holds as $x \notin U$ (Case 6 does not apply)

$$f'(x) = f(x) \geq k_3 + k_2 + (j - 1) = k'_3 + k'_2 + j.$$

The inequalities (3.3) hold in G' as the right hand side decreased by 2 and the left hand side decreased by at most 2.

Note that in all remaining cases

- (i) $|U \cap C| \leq 1$, for $C \in \mathcal{C}$. As we always have $|V(G')| \leq |V(G)| - 1$ and $k'_3 + k'_2 + k'_1 \leq k_3 + k_2 + k_1$ the inequalities (3.3) will hold for any G' .

Case 8. There is $1 \leq i \leq s$ with $S_i \cap U \neq \emptyset$.

Let i be the least i with $S_i \cap U \neq \emptyset$. Put $I = S_i \cap U$. Then $s' = s - 1$ and all other parameters remain unchanged. If $|S_i \cap U| = S_i$, we update the order of the parts in the sequence \mathcal{S} , in the following way, for $i \leq j \leq s'$ we put $S'_j = S_{j+1}$. If $|S_i \cap U| \neq S_i$ the order remains the same. Note that $|V(G')| \leq |V(G)| - 1$.

The inequalities (1) for A'_j with $1 \leq j \leq k'_1$ and (1') for S'_j with $1 \leq j < i$ hold as both sides does not change. For every vertex from parts S_{j+1}, \dots, S_s the right hand

side of the inequality (1') decreases by at least one, therefore inequalities hold. For the vertices from $S_i \setminus U$ (this set may be empty) both sides of inequality does not change, therefore inequality holds as before.

Note that in all remaining cases

- (i) Inequalities (1) and (1') will hold in any G' , provided that the right hand side does not increase.

Case 9. There is $C \in \mathcal{C}$ with $C \cap U \neq \emptyset$.

As Case 7 does not apply, $|C \cap U| = 1$. We put $I = C \cap U$. Say that $C \setminus U = \{u, v\}$ then we put $\{u, v\}$ into class \mathcal{B}' . It is straightforward that vertices from $\{u, v\}$ satisfy (2.1). They also satisfy (2.2) as

$$f'(u) + f'(v) = f(u) + f(v) \geq |V(G)| - 1 = |V(G')|.$$

Case 10. There is $B \in \mathcal{B}$ with $B \cap U \neq \emptyset$.

We put $I = B \cap U$. Say that $B \setminus U = \{u\}$. We put $\{u\}$ to the very beginning of the class \mathcal{A}' . By the observations above, all the inequalities hold, and hence G' is on-line f' -choosable (note that for Inequalities (1) and (1'), the right hand side does not increase, as k'_2 decreases by 1 and the index increases by 1).

It is easy to see that one of the 10 cases above occurs and hence G is on-line f -choosable. \square

4. GRAPHS WITH INDEPENDENCE NUMBER 3

Theorem 6. *If G is a graph with independence number at most 3 and $|V(G)| \leq 2\chi(G)$, then $\chi(G) = \text{ch}^{\text{OL}}(G)$.*

Proof of Theorem 6. Without loss of generality, we can assume that G is a complete multipartite graph with parts of size at most 3. We are going to verify that G satisfies Lemma 5 with $\mathcal{S} = \emptyset$, $f \equiv \chi(G)$ and arbitrary order on the class \mathcal{A} (when $\mathcal{S} = \emptyset$ the remaining classes of the partition are determined). Let k_1, k_2, k_3 denote the sizes of parts of sizes 1, 2, 3, respectively.

Inequalities for the single vertices (1), (2.1), (3.1) hold as $f(v) = \chi(G) = k_1 + k_2 + k_3$. Condition on pairs of vertices (2.2), (3.2) hold since $f(u) + f(v) = 2\chi(G) \geq |V(G)|$ (by the assumption on G). Moreover adding $\chi(G) = k_3 + k_2 + k_1$ on both sides of the inequality (3.2) gives (3.3).

Now, by Lemma 5 G is on-line chromatic-choosable. \square

It was shown in [19] that Conjecture 1 is true for graphs with independence number at most 3. The proof is a little complicated. Next we give an alternative proof of this result. We shall need the following lemma proved in [10] and [16].

Lemma 7. *A graph G is k -choosable if it is L -colourable for every k -list assignment L such that $|\bigcup_{v \in V} L(v)| < |V|$.*

Theorem 8. *If G is a graph with independence number at most 3 and $|V(G)| \leq 2\chi(G) + 1$, then $\chi(G) = \text{ch}(G)$.*

Proof. For a contradiction let G be a counterexample with minimum number of vertices. Let L be a $\chi(G)$ -list assignment such that G is not L -colourable. By Theorem 6, we may assume that $|V(G)| = 2\chi(G) + 1$ and by Lemma 7 we assume that the number of colours occurring on all the list is at most $2\chi(G)$.

We can also assume that for every part $\{u, v\}$ of size 2 the lists $L(u)$ and $L(v)$ are disjoint. If not, then we pick a colour $c \in L(u) \cap L(v)$ and use it to colour both vertices. The remaining graph $G' = G - \{u, v\}$ still satisfies $|V(G')| \leq 2\chi(G') + 1$. Now, if G' is chromatic-choosable then G' is colourable from $L - \{c\}$. But this would

imply that G is colourable from L . Thus G' is not chromatic-choosable which is a contradiction with the minimality of G . For the very same reason there is no colour that belongs to all three lists of vertices of any part of size 3 in G .

As $|V(G)| = 2\chi(G) + 1$ there exists at least one part of size 3 in G , say $\{u, v, w\}$. Each vertex has a list of size $\chi(G)$ and the total number of colours is at most $2\chi(G)$, therefore there exists a colour c which belongs to lists of two vertices from this part, say $c \in L(u) \cap L(w)$.

We are going to construct an L -colouring of G in two steps. First, we use c to colour u and w , remove them from G and remove colour c from all lists. Then we prove that the remaining graph $G' = G - \{u, w\}$ is on-line f' -choosable, where

$$f'(v) = \begin{cases} \chi(G) & \text{if } c \notin L(v), \\ \chi(G) - 1 & \text{if } c \in L(v). \end{cases}$$

In particular, G' can be coloured from $L - \{c\}$, which finishes the colouring of G and gives the final contradiction.

The only thing we need to verify is that G' and f' satisfy the assumptions of Lemma 5 with $\mathcal{S} = \emptyset$ and parts from \mathcal{A} ordered in such a way that the part $\{v\}$ has the greatest index. Let $k_1, k_2, k_3, k'_1, k'_2, k'_3$ denote the numbers of parts of size 1, 2 and 3 in G and G' , respectively. We have

$$k'_1 = k_1 + 1, \quad k'_2 = k_2, \quad k'_3 = k_3 - 1.$$

Inequalities (2.1), (3.1) hold as for any x in part of size 2 or 3 in G'

$$f'(x) \geq \chi(G) - 1 \geq k_3 + k_2 - 1 = k'_3 + k'_2$$

The part of size 1, say $\{x\}$, with index less than k'_1 satisfies (1) as

$$f'(x) \geq \chi(G) - 1 = k'_3 + k'_2 + k'_1 - 1.$$

The remaining part of size 1, namely $\{v\}$, satisfies (1) as $f(v) = \chi(G) = \chi(G')$ (as $c \notin L(v)$). Inequalities (2.2) hold since colour c belongs to the list of at most one vertex in every part of size 2 in G' . Therefore, for any $\{x, y\}$ part of size 2 in G' we have

$$f'(x) + f'(y) \geq 2\chi(G) - 1 = |V(G')| - 1.$$

It remains to verify inequalities (3.2) and (3.3). Let x, y, z be any three vertices forming a part of size 3 in G' . Then

$$f'(x) + f'(y) \geq 2\chi(G) - 2 = |V(G')| - 1,$$

$$f'(x) + f'(y) + f'(z) \geq 3\chi(G) - 2 = |V(G')| - 1 + k'_3 + k'_2 + k'_1.$$

The latter inequality follows from the fact c is not in all three $L(x), L(y), L(z)$. \square

5. THE COMPLETE MULTIPARTITE GRAPH $K_{3\star k}$

There are not many graphs for which the exact value of their choice numbers are known. The graphs $K_{3\star k}$ are among those few graphs G for which $\text{ch}(G)$ are determined. In [10], Kierstead proved that $\text{ch}(K_{3\star k}) = \lceil (4k-1)/3 \rceil$. In this section, we present an alternative proof of this result.

Theorem 9 (Kierstead 2000). *For any positive integer k , $\text{ch}(K_{3\star k}) = \lceil \frac{4k-1}{3} \rceil$.*

The lower bound $\text{ch}(K_{3\star k}) \geq \lceil \frac{4k-1}{3} \rceil$ was given by Erdős, Rubin and Taylor [3]. As the proof is very short, we include it here for the convenience of the reader. Let $q = \lceil \frac{4k-1}{3} \rceil - 1$. Let A, B, C be disjoint colour sets with $|A| = \lfloor q/2 \rfloor$ and $|B| = |C| = \lceil q/2 \rceil$. Assume the parts of $K_{3\star k}$ are $V_i = \{x_i, y_i, z_i\}$ for $i = 1, 2, \dots, k$. Let $L(x_i) = A \cup B, L(y_i) = B \cup C$ and $L(z_i) = A \cup C$. Then $|L(v)| \geq q$ for each vertex v , and if f is an L -colouring of $K_{3\star k}$, then f uses at least 2 colours on V_i , and hence the total number of used colours is at least $2k$. However, straightforward

calculation shows that $|A \cup B \cup C| \leq 2k - 1$. Therefore $K_{3\star k}$ is not L -colourable and hence $\text{ch}(K_{3\star k}) \geq q + 1 = \lceil \frac{4k-1}{3} \rceil$.

The inequality $\text{ch}(K_{3\star k}) \leq \lceil \frac{4k-1}{3} \rceil$ is a straightforward consequence of the following lemma.

Lemma 10. *Let G be a complete multipartite graph with parts of size 1 and 3. Let $\mathcal{A}, \mathcal{S}, \mathcal{C}$ be a partition of the set of parts of G into classes such that \mathcal{A} and \mathcal{S} contains only parts of size 1, while \mathcal{C} contains all parts of size 3. Let k_1, s, k_3 denote the cardinalities of classes $\mathcal{A}, \mathcal{S}, \mathcal{C}$, respectively. Suppose that class \mathcal{A} and \mathcal{S} are ordered, i.e. $\mathcal{A} = (A_1, \dots, A_{k_1})$ and $\mathcal{S} = (S_1, \dots, S_s)$. If $f : V(G) \rightarrow \mathbb{N}$ is a function for which the following conditions hold*

$$f(v) \geq k_3 + i, \quad \text{for all } 1 \leq i \leq k_1 \text{ and } v \in A_i \quad (1)$$

$$f(v) \geq 2k_3 + k_1 + i, \quad \text{for all } 1 \leq i \leq s \text{ and } v \in S_i \quad (1')$$

$$f(v) \geq k_3, \quad \text{for all } v \in C \in \mathcal{C} \quad (3.1)$$

$$f(u) + f(v) \geq 2k_3 + k_1, \quad \text{for all } u, v \in C \in \mathcal{C} \quad (3.2)$$

$$\sum_{v \in C} f(v) \geq 4k_3 + 2k_1 + s - 1, \quad \text{for all } C \in \mathcal{C}, \quad (3.3)$$

then G is f -choosable.

Proof. Assume the lemma is not true. Let G be a multipartite graph with parts divided into $\mathcal{A}, \mathcal{S}, \mathcal{C}$, and let f be a function fulfilling the inequalities (1)-(3.3) while G is not f -choosable. Moreover, suppose G is a counterexample with the minimum possible number of vertices. By Lemma 7 there exists a list assignment $\{L(v)\}_{v \in V(G)}$ with each $|L(v)| = f(v)$ and $|\bigcup_{v \in V(G)} L(v)| \leq |V(G)| - 1 = 3k_3 + k_1 + s - 1$ such that G is not L -colourable.

The claims below prove a series of properties of G and list assignment L . In the arguments we often make use the minimality of G and consider some smaller graphs with modified list assignment. The modified graph will be denoted by G' and, unless otherwise stated, the classes of its vertices $\mathcal{A}', \mathcal{S}'$ and \mathcal{C}' , together with orders on \mathcal{A}' and \mathcal{S}' , are inherited from G . The parameters k'_1, s', k'_3 correspond to the analogous parameters of G' . The modified list assignment is going to be denoted by $L'(v)$ and $f'(v) = |L'(v)|$ for all $v \in V(G')$.

Claim 0. For any $C \in \mathcal{C}$ we have $\bigcap_{v \in C} L(v) = \emptyset$.

Proof. Suppose there is $C \in \mathcal{C}$ with $c \in \bigcap_{v \in C} L(v)$. We colour all vertices of C with c and consider the smaller graph $G' = G - C$ with list assignment $L'(v) = L(v) - \{c\}$. It is easy to verify that G' (with $\mathcal{A}', \mathcal{S}', \mathcal{C}'$ inherited from G) and f' satisfies the assumptions of the lemma. By the minimality of G , G' is L' -colourable. This implies that G is L -colourable, in contrary to our assumption. \square

Claim 1. For any $u, v \in C \in \mathcal{C}$ if $f(u) + f(v) = 2k_3 + k_1$, then $L(u) \cap L(v) = \emptyset$.

Proof. Suppose that for some part $C = \{u, v, w\}$ we have $f(u) + f(v) = 2k_3 + k_1$ and there exist $c \in L(u) \cap L(v)$. Then we colour u and v with c , and consider the smaller graph $G' = G - \{u, v\}$ with lists $L'(x) = L(x) - \{c\}$ for all $x \in V(G')$. The partition $\mathcal{A}', \mathcal{C}'$ is inherited from G and $\mathcal{S}' = (\{w\}, S_1, \dots, S_s)$ has one more part, namely $\{w\}$, while all other parts have shifted index, i.e., $S'_{i+1} = S_i$ for $1 \leq i \leq s$. In particular, $k'_1 = k_1$, $s' = s + 1$, $k'_3 = k_3 - 1$. Note that the inequality (1') holds for $S'_1 = \{w\}$ as

$$f'(w) = f(w) \geq (4k_3 + 2k_1 + s - 1) - (2k_3 + k_1) = 2k_3 + k_1 + s - 1 = 2k'_3 + k'_1 + 1,$$

and (1') holds for $S'_{i+1} = S_i = \{x\}$ for $1 \leq i \leq s$ as

$$f'(x) \geq f(x) - 1 \geq (2k_3 + k_1 + i) - 1 = 2k'_3 + k'_1 + i + 1.$$

Again, it is easy to verify that G' with f' satisfies the assumptions of the lemma. Hence G' is L' -colourable, implying that G is L -colourable, a contradiction. \square

Claim 2. For any $v \in C \in \mathcal{C}$ we have $f(v) > k_3$, i.e., the inequality (3.1) is not tight.

Proof. In order to get a contradiction suppose that $\{v, u, w\} = C \in \mathcal{C}$ and $f(v) = k_3$. We separate the argument into two cases:

- * $L(v) \cap (L(u) \cup L(w)) \neq \emptyset$. Without loss of generality, assume that $L(v) \cap L(u) \neq \emptyset$. Let $c \in L(v) \cap L(u)$. We colour u and v with c , and consider the smaller graph $G' = G - \{u, v\}$ with lists $L'(x) = L(x) - \{c\}$ for all $x \in V(G')$. The partition \mathcal{S}' , \mathcal{C}' is inherited from G and $\mathcal{A}' = (A_1, \dots, A_{k_1}, \{w\})$ has one more part, namely $\{w\}$, appended to the inherited ordering. In particular, $k'_1 = k_1 + 1$, $s' = s$, $k'_3 = k_3 - 1$. Note that the inequality (1) holds for $A'_{k'_1} = \{w\}$ as

$$f'(w) = f(w) > (2k_3 + k_1) - k_3 = k'_3 + k'_1.$$

Let $x, y \in C \in \mathcal{C}'$. Inequality (3.2) for x and y hold as either $f(x) + f(y) > 2k_3 + k_1$ and therefore

$$f'(x) + f'(y) \geq f(x) + f(y) - 2 > 2k_3 + k_1 - 2 = 2k'_3 + k'_1 - 1,$$

or $f(x) + f(y) = 2k_3 + k_1$ and therefore by Claim 1 $L(x)$ and $L(y)$ are disjoint.

$$f'(x) + f'(y) \geq f(x) + f(y) - 1 = 2k_3 + k_1 - 1 = 2k'_3 + k'_1.$$

With these observations, it is easy to verify that G' with f' satisfies the assumptions of the lemma. Hence G' is L' -colourable and therefore G would be L -colourable, a contradiction.

- * $L(v) \cap (L(u) \cup L(w)) = \emptyset$. Then by (3.3) and our assumption $f(v) = k_3$ we get that

$$f(u) + f(w) \geq (4k_3 + 2k_1 + s - 1) - k_3 = 3k_3 + 2k_1 + s - 1.$$

On the other hand the total number of colours is at most $3k_3 + k_1 + s - 1$ and as $L(v)$ is disjoint with $L(u) \cup L(w)$ we get $|L(u) \cup L(w)| \leq 2k_3 + k_1 + s - 1$. Combining the two inequalities above we obtain

$$|L(u) \cap L(w)| \geq k_3 + k_1.$$

We colour vertex v by any colour $c \in L(v)$. Then we consider graph $G' = G - \{v, u, w\} + \{x\}$, where x is a brand new vertex which is convenient to be seen as a merger of u and w . Let $L'(y) = L(y) - \{c\}$ for all $y \in V(G') - \{x\}$ and $L'(x) = L(u) \cap L(w)$. The partition \mathcal{S}' , \mathcal{C}' is inherited from G and $\mathcal{A}' = (A_1, \dots, A_{k_1}, \{x\})$ has one more part, namely $\{x\}$, appended to the inherited ordering. In particular, $k'_1 = k_1 + 1$, $s' = s$, $k'_3 = k_3 - 1$. Note that the inequality (1) holds for $A'_{k'_1} = \{x\}$ as

$$f'(x) = |L(u) \cap L(w)| \geq k_3 + k_1 = k'_3 + k'_1.$$

The other inequalities for G' and f' hold for the same reasons as before. So G' is L' -colourable. We obtain an L -colouring of G , by colouring the vertices u and w with the colour of x and colouring v with c , a contradiction. \square

Claim 3. $k_1 = 0$.

Proof. Suppose that $k_1 \neq 0$. Then let $A_1 = \{v\}$. We colour v with any colour $c \in L(v)$ and consider the smaller graph $G' = G - \{v\}$ with lists $L'(x) = L(x) - \{c\}$ for all $x \in V(G')$. The partition $\mathcal{A}' = (A_2, \dots, A_{k_1})$, \mathcal{S}' , \mathcal{C}' is inherited from G .

Note that \mathcal{A}' has one part less and $k'_1 = k_1 - 1$, $s' = s$, $k'_3 = k_3$. Now, we verify the inequalities (1)-(3.3) for G' and f' :

- * (1) holds as the indices of parts are decreased, i.e. $A'_i = A_{i+1}$ for $1 \leq i < k'_1$;
- * (1') holds as k_1 decreases,
- * (3.1) holds as, by Claim 2, it is not tight in G ,
- * (3.2) holds for $x, y \in C \in \mathcal{C}'$ as k_1 decreases and either (3.2) is not tight for u, v in G , or $f'(x) + f'(y) \geq f(x) + f(y) - 1$ (by Claim 1);
- * (3.3) holds as k_1 decreases by 1 and the left hand side decreases by at most 2 (by Claim 0).

Once again by minimality of G we get that G' is f' -choosable, and that gives that G is L -colourable, a contradiction. \square

We are now ready to derive the final contradiction. If $k_3 = 0$ then G has only parts of size 1 in \mathcal{S} and it is immediate that G is f -choosable. Assume $k_3 \neq 0$. Recall that the total number of colors in all lists is at most $3k_3 + s - 1$. Let $\{u, v, w\}$ be a part of size 3. Then $f(u) + f(v) + f(w) \geq 4k_3 + s - 1 > 3k_3 + s - 1$ and therefore there must be a colour c which appears in two out of three colour sets $L(u)$, $L(v)$, $L(w)$, say $c \in L(u) \cap L(v)$.

We colour u and v with c and consider $G' = G - \{u, v\}$ with lists $L'(x) = L(x) - \{c\}$. Again, the partition $\mathcal{S}', \mathcal{C}'$ is inherited from G and we simply put $\mathcal{A}' = (\{w\})$. Thus, $k'_1 = 1$, $s' = s$, $k'_3 = k_3 - 1$. We verify the inequalities (1)-(3.3) for G' with f' . The inequality (1) for $A'_1 = \{w\}$ holds as

$$f'(w) = f(w) > k_3 = k'_3 + 1.$$

All the other inequalities hold for analogous reasons as before. Once again, by minimality of G , we get that G' is f' -choosable, and that gives that G is L -colourable, a contradiction. \square

The last result of the paper is another immediate consequence of Lemma 5.

Corollary 11. $\text{ch}^{\text{OL}}(K_{3\star k}) \leq \frac{3}{2}k$, for any positive integer k .

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